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SOME NEW RATIOS OF CONIC CURVES.

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Every conic curve is equidistant from a *fixed point* or focus [S or f], and a *fixed circle*, the *director circle*, whose center is the other, or second focus to the curve [S'], and whose radius [$S'D$] is equal to the major axis [AA' or aa', Fig. 1]. For, in the ellipse, $S'p+pf=aa'=S'D=S'p+pE$; while in the hyperbola, $S'P-PS=AA'=S'D=S'P-PE$. So that in both curves alike $pf=pE$, or $PS=PE$, for all possible positions of p or P.

In the parabola, since one focus [S'] is at infinity, the infinite director circle, whose center it is, must coincide with the usual rectilinear directrix to the curve. And since in every conic curve there are two precisely similar foci, either both real, or one real, and one ideal at infinity, either one of these can therefore be taken as the center of our director circle; the other focus thereby becoming our fixed point. But note: in the hyperbola, when that focus [S'], which lies within the opposite branch of the curve, is taken for our fixed point, and its director circle, as a consequence, lies partly or wholly within that branch which we are determining, then the equidistance of P must be measured, not along the shorter segment of the diameter, or produced diameter of the circle through P; but along its greater segment—*e. g.* in Fig. 1, QS is equal to QI, not to QK; and $P'S$ to PJ. And furthermore, when the distance SD of the fixed point from its director circle is twice, or greater than twice the radius of that circle—or in other words, when $S'S$ the inter-focal distance is thrice, or more than thrice AA' the major circle,—then the said circle cannot cut, but must lie wholly within

the opposite branch of the hyperbola. We may remark in passing that the term director circle, in place of ever meaning, as it undoubtedly should, this curvilinear directrix, is frequently misapplied to what is more properly called the *orthocycle;* which latter is in no sense a directrix; and merely happens to coincide with it, and the true director circle, in the special case of the parabola.

We can, still further, describe two unequal circles. The smaller, with its center at the fixed point, f or S, and with a radius which is less than half aa' or AA', the major axis. And the greater, with its center at S', concentric with the director circle; while its radius is less, or greater than that of said director circle, by the radius of the smaller circle, according as f or S was the point chosen for that smaller circle's center. In which case the ellipse or hyperbola, as the case may be, will evidently lie equidistant from the circumferences of the two unequal circles. So that any conic curve can also be generated by the equidistance between two such circles, which, in the special case of the parabola, become a circle and a right line.

Returning to the director circle and its fixed point, we may trace the sequence of the generated curves as follows. Commencing with the circle, wherein the fixed point and the center of its director circle are one, we pas immediately into the ellipse, as soon as said fixed point and center separate. The ellipse becoming flatter and flatter as the fixed point moves towards the circumference of its circle. Nor does the direction of this approach make any difference; since each fixed point, or focus, must have, on the opposite side of the bifocal ellipse, its precisely similar twin focus, and center of its director circle. Flatter and narrower grows the ellipse, until the point having approached infinitesimally close to the circumference—or, what amounts to the same thing, the center of the director circle having become, proportionately, infinitely distant from its fixed point—our ellipse passes, momentarily, into a pair of parabolas. But the instant that the "becoming" of the infinitesimal approach ceases, and the point is now actually *on* the circumference, then these momentary parabolas vanish; and the two sides of our degenerated conic coalesce with its major axis into one horizontal right line, running from said fixed point on the circumference in either direction. Said circumference of the director circle being now represented by a perpendicular right line through the point; since it is, proportionately, infinite in radius. The infinitesimal progress reappearing, as our point crosses the circumference [now, once more, of finite radius] and recedes infinitesimally away from it, this degenerated line [extended, now, externally along the major axis] splits; first, for an instant, again into a second pair of parabolas, and then into a narrow hyperbola. Said hyperbola becoming broader and broader, and its two opposite branches, proportionately, closer and closer, as the fixed point recedes from its director circle. Until point and circle having become infinitely distant, and therefore the radius $S'D$, which is equal to the major axis AA , proportionately infinitesimal, the two opposite branches of our hyperbola, fusing into one, have degenerated once more into a right line, but now lying at right angles to the major axis. And similarly, while our point had as yet moved but

an infinitesimal distance from the circumference, then the conjugate curve to the resultant extremely narrow hyperbola, or pair of parabolas, being extremely broad, must have a major axis and director circle of infinitesimal magnitude; strictly corresponding, then, in its resultant ratios to one immeasurable distant. While as the fixed point of the original curve recedes from its circle, so reciprocally do the director circles of its conjugate curve grow in proportionate magnitude; and the comparative distance of their respective points decrease. Until, said fixed point of the parent curve having passed to infinity, and its hyperbola therefore, as already stated, degenerated into a right line, perpendicular to the major axis, the conjugate curve must now possess director circles infinite in radius, and infinitesimally close to their respective fixed points; and thus be, first, momentarily, a pair of parabolas, and then a similar pair of degenerated right lines, lying parallel to, and indeed inside of, and coalescing with, the degenerated original curve. At the one extreme, then, of this sequence that we have traced, lies the circle, with its foci coalesced into the central pole. And at the opposite extreme, as the farthest possible removed form, lies the degenerated "perpendicular right line hyperbola," with its infinitely distant fixed point on the polar of said pole. While midway lie the two sets of parabolas, facing in opposite directions, and each flashing momentarily into being, as the ellipse sinks into, or the hyperbola arises from their medial point of indifference; which is, again, a degenerated horizontal right line.

From the above the following theorems can be deduced:

Theorem 1. Even as the sum, in the ellipse, or difference, in the hyperbola, of the two focal distances of any point P on the curve is constant, and equal to the major axis; so, similarly, is the sum or difference of its distances from the two director circles.

Theorem 2. The line SE or fE, SH or fH, etc., joining the fixed point S or f to the extremity of any radius $S'E$ or $S'H$ of its director circle, is ever bisected by the auxiliary circle to the curve. While if the two radii, in the ellipse, or diameters in the hyperbola, of the two director circles be drawn through the same point P or p on the curve, and their extremities joined with their respective fixed points [S or f, as the case may be]; and the bisections of such two lines, by the auxiliary circle, be joined by a right line, then will this line be the tangent at P or p.

Theorem 3. In the ellipse, the produced latus rectum, which is the double sine through f, the fixed point, will intersect upon the director circle those two radii, which determine the minor axis of the curve. For if $SB=Bf=BH$ [Figure 1], then $S'fH$ are concyclic; $S'fH$ a right angle; and thus fH the produced semi-latus rectum.

Theorem 4. While correspondingly, in the hyperbola, those two diameters of the director circle, whose respective tangents pass through the fixed point, are parallel to the asymptotes, determining at infinity the ideal minor axis of the curve. Which two tangents, furthermore, are perpendicularly bisected by those asymptotes, at their common intersection with the rectilinear directrix.

For, let $S'H$ be the radius, whose tangent HS passes through S, the fixed point [Figure 1]. And let ZCZ'', $Z'CZ$ be the asymptotes, meeting the directrix in YY'. Then $S'H:S'S=AA':SS'=CA:CS=CY:CS$. So that the right-angled triangles $S'HS$ and CYS are similar, with $S'H$ parallel to the asymptote CYZ; and SY half of HS, even as CS is of $S'S$. And thus CYZ bisects SH at right angles.

Corollary 1. If a perpendicular Cb were raised at C the center of the curve, and through either of the apeci A or A' a line Ab or $A'b$ be drawn, parallel to either $S'H$, or an asymptote, it will cut Cb in b, an extremity of the accepted minor axis of the hyperbola—the major axis of its conjugate.

Theorem 5. If from any point, say K, upon the circumference of the director circle, two tangents KI and KG be drawn to the curve, cutting the circle again in GI, and these points GI be joined; then will said GI be a third tangent to the curve. And the resultant circumscribing triangle KGI will have for its orthocenter S [or f, as the case may be], the fixed point.

Taking the hyperbola [Figure 1]. There are evidently innumerable circumscribing triangles possible. And similarly, innumerable triangles, having S, the fixed point, for their orthocenter. And finally, innumerable triangles which fulfill both of these conditions.

Let one of these latter, KGI, be drawn, both circumscribing the hyperbola, and with its orthocenter at S. We will prove that its vertices KGI lie on the director circle.

The auxiliary circle to the curve cuts the three tangents KI, KG, and GI in points vi, wk, and ug, respectively. Then, by a well known theorem, SiI, SkK, and SgI are right angles; and thus gki are the pedal points of the perpendiculars from the vertices upon the sides of the triangle KGI. And therefore the auxiliary circle, as the circumscribing circle to this pedal triangle kgi, is the "nine points circle" of triangle KGI; having its center C collinear with, and bisecting the distance between the orthocenter S and circumcenter S' of KGI; while its radius is half that of the circumscribing circle, since it is itself the circumscribing circle to the medial triangle uvw. So that since $CS=CS'$, and $CA=\frac{1}{2}AA'=\frac{1}{2}S'D$, therefore the director circle is the circumscribing circle to the triangle KGI, tangential to, and thus circumscribing the hyperbola, and with its orthocenter at the fixed point S.

An additional proof is as follows. If uvw be the points where the sides of the circumscribing triangle KGI are cut by its "nine points circle"—the auxiliary circle,—then such points thereby bisect those sides; and the perpendiculars $S'u$, $S'v$, and $S'w$ upon them must thus meet in S', the circumcenter to the triangle, and second focus to the curve.

And since the same proof holds true, and the auxiliary circle is the "nine points circle" for any triangle, which is both tangential to the hyperbola, and has its orthocenter at S; then, conversely, any two tangents from any point upon the circumference of the director circle must determine a third tangent to the curve; and the resultant circumscribing triangle have its orthocenter at S, the fixed point.

In a like manner, if a triangle be drawn circumscribing an ellipse, and with its orthocenter at f, the fixed point of the curve; then the auxiliary circle is still the "nine points circle," and thus the director circle its circumscribing circle.

In the parabola, since the director circle, being infinite in radius, has become the rectilinear directrix, no real circumscribing triangle, having its orthocenter at the focus S, can be drawn. Although the tangent on the vertex, which represents, of course, the auxiliary circle, would also be the infinite "nine points circle" to any such triangle; since it passes through the pedal points of the perpendiculars from the orthocenter S upon the tangents.

Corollary 1. The auxiliary circle, as the "nine points circle," bisects not only the sides KG, KI, and GI of the circumscribing triangle; but also the three lines SK, SG, and SI, joining its vertices to its orthocenter. While, if the perpendiculars Sg, Sk, and Si be produced, they will ever meet their circumcircle, the director circle, at double the distances Sg, Sk, and Si, respectively.

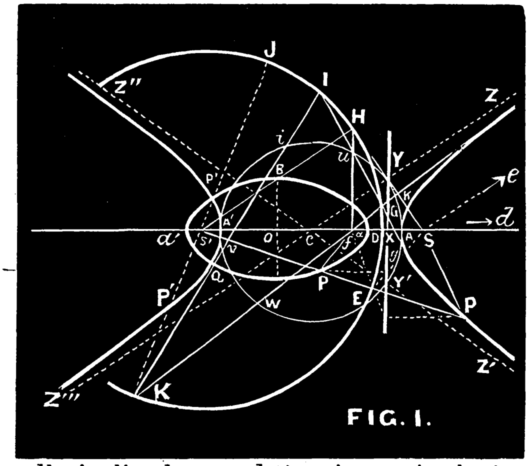

FIG. 1.

Corollary 2. The "nine points circle" is thus tangential, not only to the inscribed and escribed circles of triangle KGI, but also to its inscribed ellipse, or escribed hyperbola.

Corollary 3. If the pedal triangle kgi be drawn, then, by known theorems, its sides must be equally inclined to, and thus its angles be internally or externally bisected by the sides of its parent triangle KGI; whose three apeci K, G, and I, and orthocenter S, furthermore, must ever be the centers of its inscribed and escribed circles. While points SkI, SGi and SgK, like KGk, must be ever collinear, in ellipse, or hyperbola. So that points SGI, SGK, and SKI must ever lie on the arcs of circles, whose radii are equal to each other, and to that of the director circle—namely, to the major axis.

But the most important deductions from the foregoing are the following theorems; by means of which the exact opposite and reciprocal reverse, point for point, of any conic curve can be precisely determined. A parabola reversing into an equal, but opposite parabola; a circle into a perpendicular right line; and an ellipse into an exactly reciprocal hyperbola; and conversely; with equal major axi.

Theorem 6. If an ellipse and a hyperbola, with major axi lying collinearly along one right line, and with a common director circle [and thus also with equal, though not coincident, major axi], have their respective fixed points each on the polar of the other, in respect to their common circle, *i. e.*, [Figure 1] $S'f:S'D=S'D:S'S$; then said ellipse and hyperbola will be each the reverse or

reciprocal of the other. The common radii of their common director circle determining corresponding points upon each of the two curves, with focal distances which are equally inclined to their respective major axi.

Thus, if $S'pEP$ be a common radius of the common director circle, cutting and determining the ellipse in p, and the hyperbola in P, then will angles pfS', and PSd, or pfa and PSA be equal. Similarly, the same common radius will determine the lati recti in both curves. Or finally, that radius $S'H$, lying parallel to an asymptote of the hyperbola [Theorem 4], thus determining at infinity its ideal minor axis, determines also the minor axis of the reciprocal ellipse [Theorem 3]; and subtends an equal angle in both curves; BfS' of the one equalling eSd of the other. And so on, for all the radii.

Taking first the radius $S'H$ parallel to the asymptote ZCZ''. Then by construction, $S'f{:}S'H = S'H{:}S'S$; so that the triangles $S'HS$ and $S'fH$ are similar right angled triangles; and fH is thus the produced semi-latus rectum of the ellipse. Therefore $S'H$ must pass through and determine B, an extremity of the minor axis of the ellipse [Theorem 3]; even as it also meets the asymptote of the hyperbola at infinity. While by construction, angles $HS'f$, ZCd, and eSd are equal, and thus also BfS' and eSd.

Next, taking any other radius $S'pEP$; determining the ellipse in p, and the hyperbola in P. Join pf, PS, Ef, and ES. Then again $S'f{:}S'E = S'E{:}S'S$; and so $S'ES$ and $S'fE$ are similar triangles, with equal angles opposite to homologous sides, $S'ES$ to $S'fE$, and $S'Ef$ to $S'SE$. But pE being equal to pf, and PE to PS, angles pEf and pfE are equal; and also PES to PSE. And therefore angles $pES - pEf$, or fES equal $S'fE - pfE$, or pfS'. While on the other hand, angles $S'SE + PSE$, or PSS' equal $S'Ef + PES$. So that their supplementary angles PSd and fES are equal; and therefore $pfS' = PSd$, and $pfa = PSA$.

And similarly, for any radius of the common director circle. So that if the one be taken which determines fl, the semi-latus rectum in the ellipse, and cuts the hyperbola, say, in L; then angle LSd can be shown equal to the right angle lfS'; and thus LS be the semi-latus rectum of the hyperbola.

Said ellipse and hyperbola are thus reciprocal curvilinear forms; or the exact reversals and opposites of each other.

But note: when a radius beyond $S'H$ is taken, as *e. g.* $KS'J$, determining therefore upon its *diameter* the reciprocal points p' on the ellipse, and P' on the other branch of the hyperbola, then the axial angle of P', which is equal to $p'fS'$ and fJS is, not $P'Sd$, but $P'SS'$. So that the axial angle, in the hyperbola, must thus in every case be measured towards its ideal minor axis, as it is towards the real, in the ellipse.

If in place of f and S, we choose S' for our focal angle, then our theorem still holds true; for obviously, $PS'd = pS'f$, and $P'S'A' = p'S'a'$, etc.

Applying this theorem to the parabola, since it can be considered either an extreme ellipse, or an extreme hyperbola, our reciprocal curves will take the form of two precisely similar parabolas, facing in opposite directions a common directrix. In which case, manifestly the above theorem holds true. While if

we take a circle as our ellipse: then since $S'f$ its inter-focal magnitude equals zero, its polar to f must lie at infinity. And thus its reciprocal hyperbola, or reverse curvilinear form, whose fixed point S is on that polar, must assume, as already pointed out, the degenerate shape of a right line, lying at infinity, perpendicular to the collinear major axi.

If, then, we draw about point S' two concentric circles; the larger with double the radius of the smaller, and representing its director circle, and let them be cut in p and E respectively by a radius $S'pE$. Then $S'pE$ produced will represent the direction in which, at infinity, lies the ideal point P upon this degenerated right line hyperbola. While any other radius $S'aD$, making less than a right angle with $S'pE$, may represent the collinear major axi; and thus a line E.........S, through E, parallel to $S'aD$, will represent the direction at infinity of the ideal fixed point S of this degenerated hyperbola. Then, by hypothesis, $PE=PS$; and thus the ideal angles PSE, PSa, or PSA equal the real angles PES and $pS'a$. So that our theorem is again true.

Corollary 1. Thus if any points p and P upon the reciprocal curves subtend equal angles pfS' and PSd, or pfa and PSA; then they are thereby corresponding points, lying on one common determining radius or diameter.

Corollary 2. If pm and PM, their perpendiculars to the directrix, be drawn, then angles fpm and SPM are equal.

Corollary 3. If the lines fE and SE, or fJ and SJ, joining the respective fixed points to the extremity E, or J, of the common determining radius $S'E$, or diameter $P'S'p'J$, be each bisected in r and R by the respective auxiliary circles [Theorem 2], then the line rR, joining said medial points, is thereby fixed, both in direction, and magnitude; being parallel to, and one half of, fS.

Theorem 7. The said reciprocal curves, ellipse and hyperbola, having a common director circle, have thereby also both reciprocal eccentric ratios $S'f:aa'$, or $fa:aX$ of the one, equaling $AA':S'S$, or $AX:SA$ of the other; and a common rectilinear directrix YXY'. Which, furthermore, bisects the reciprocal focal distance, *i. e.*, fS. So that $fa:aX$ of the ellipse is not only of equal ratio to its reciprocal $AX:SA$ of the hyperbola; but is also of equal magnitude, fa equalling AX, and aX equalling SA [Figure 1].

For, bisecting fS in X by the perpendicular YXY', then $fD:DS=fD:fS-fD=fS-DS:DS$, and thus their halves are in a similar proportion; $fa:fX-fa=XS-AS:AS$; or $fa:aX=AX:SA$. While $fa+aX=XA+AS$; and so $fa=AX$, and $aX=SA$.

Lastly, if O be the middle point of $S'f$ in the ellipse, and C of $S'S$ in the hyperbola, then will $fa:aX=fD:DS=S'f:S'D=S'f:aa'=Of:Oa=Of+fa:Oa+aX=Oa:OX$. So that X is on the polar of f in the ellipse; and thus the polar YXY' is the rectilinear directrix.

And similarly, in the hyperbola, $SA:AX=SD:Df=S'S:S'D=S'S=A'A=CS:CA=CS-SA:CA-AX=CA:CX$. So that YXY', as the polar of S, is the rectilinear directrix of the hyperbola.

To draw reciprocal conic curves, then, we may either, with a common

director circle, place their respective fixed points each on the polar of the other; or else, using a common rectilinear directrix, we may make their eccentric ratios reciprocally equal, both in proportion, and in size. The said eccentric ratio fa: $aX = AX : SA$, or polar proportion $fD : DS = S'f : S'D = S'D : S'S$ being the same in both cases. While in either case, also, the major axi are both collinear, and equal; and the second focus S' is in common.

Corollary 1. If the focal distance pf of a point p on one of these curves equals in magnitude the distance PM from the directrix of point P on its reciprocal curve, then conversely, must the directrix distance pm of the first point p equal the focal distance PS of the second, P.

Corollary 2. Since $SY = YH$ [Theorem 4], and $S'B = Bf = BH$, therefore $YX = OB$. And thus YY', the portion of the directrix cut off between the asymptotes of the hyperbola, ever equals the minor axis of the reciprocal ellipse.

Corollary 3. fX or SX, being the half of fS, equals the distance OC between the respective centers of the curves. For $S'S - S'f = fS$; and thus $\frac{1}{2}S'S - \frac{1}{2}S'f = \frac{1}{2}fS = fX = SX = S'C - S'O = OC$. And this distance, again, must equal the space between the several apeci, Aa, or $A'a'$. Since $fa = AX$, and $aX = SA$, and thus $fa + aX = fX = AX + aX = Aa = fa + SA = S'a' + S'A' = A'a'$.

Theorem 8. If D be the point where the director circle cuts the major axis of a conic; $S'E$ or $S'J$ any radius or diameter of this circle, determining a point P or p on the curve; DE a line joining points D and E; and ES or Ef a line from E to the fixed point S or f. Then will angle DES or DEf, as the case may be, ever be one half of the axial angle of P or p; which, in the ellipse is angle pfS', but in the hyperbola PSd, or PSS' [Theorem 6], according to the branch which point P may be upon [Figure 1].

For in Theorem 6 angles fES, pfS' and PSd were shown to be equal; while $S'fE$ and $S'ES$ being similar triangles, the homologous sides $fE : ES = S'f : S'E = S'E : S'S = S'D - S'f : S'S - S'D = fD : DS$. So that DE must bisect angle fES [Euclid VI, 3]; and thus $DEf = \frac{1}{2}pfS'$; and $DES = \frac{1}{2}PSd$.

Were $P'S'p'J$ the determining diameter chosen, then in a like manner it can be shown that $fJ : JS = fD : DS$; and thus angles $DJf = DJS = \frac{1}{2}fJS = \frac{1}{2}p'fS' = \frac{1}{2}P'SS'$.

In the parabola, PM, the perpendicular through P to the directrix, obviously represents pE or PE as above; and since angles XSM, SMP, and PSM are all equal, SM must bisect angle XSP; and thus SMX, the complement to SMP, be half of PSd, the supplement to XSP.

While, in the circle, making the same construction as in Theorem 6, angles $S'ED$, $S'DE$, and DES are all equal; and thus DES' or DES is one half of the supplement to $p'S'a$, or to the ideal angle PSa.